Program Semantics

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Why Semantics

- Formally describe the meaning (semantics) of a program (e.g., how it executes)
- Uses of formal semantics:
  - Precise definitions
  - Proofs of correctness (soudness), termination
- In practice
  - Difficult to define for a full language
  - Mostly applies to a "core" of a language

The WHILE language

Category	Domain	Meta variable
Numbers	$Z = \{0, 1, -1, \dots\}$	z
Truth values	$B = \{T, F\}$	t
Variables	$Var = \{x, y, ...\}$	x
Arithmetic expressions	AExp	a
Boolean expressions	BExp	b
Commands (statements)	Cmd	c

Context-Free Grammar of WHILE

$$
\begin{aligned}
a &::= \quad z \mid x \mid a1 + a2 \mid a1 - a2 \mid a1 * a2 \in \textsf{AExp} \\
b &::= \quad t \mid a1 \equiv a2 \mid a1 > a2 \mid \neg b \mid b1 \wedge b2 \mid b1 \vee b2 \in \textsf{BExp} \\
c &::= \quad \textsf{skip} \mid x := a \mid \\
&\qquad \textsf{if } b \textsf{ then } c1 \textsf{ else } c2 \textsf{ end} \mid \\
&\qquad \textsf{while } b \textsf{ do } c \textsf{ end} \in \textsf{Cmd} \mid \\
&\qquad c1; c2
\end{aligned}
$$

Big-Step Operational Semantics

- Shows the effect of *each statement* on the program *state*
- Contains evaluation rule for each statement
- Depends on an state (environment) $E : \mathsf{Var} \mapsto \mathsf{Value}$

$\{\}$
```
x := 6;
```
$\{x \mapsto 6\}$
```
y := 7;
```
$\{x \mapsto 6, y \mapsto 7\}$

Evaluation Rules (E-Rules)

$$\frac{\mathsf{Premise}_1 \quad \ldots \quad \mathsf{Premise}_k}{\mathsf{Conclusion}}$$

Read: if premises, then conclusion

E-Val

$$\frac{\text{True}}{E \vdash n \Downarrow \mathbf{n}}$$

Examples

- No Premises, i.e., Axioms
- $\overline{E \vdash 5 \Downarrow \mathbf{5}}$, $\overline{E \vdash -1 \Downarrow \mathbf{\text{-}1}}$
- $\overline{E \vdash \mathrm{T} \Downarrow \mathbf{True}}$, $\overline{E \vdash \mathrm{F} \Downarrow \mathbf{False}}$

E-Var

given $E = \{x \mapsto n\}$

$$\frac{E\{x\} = n}{E \vdash x \Downarrow \mathbf{n}}$$

Examples

- $\dfrac{E\{x\} = 5}{E \vdash x \Downarrow \mathbf{5}}$
- $\dfrac{E\{y\} = -1}{E \vdash y \Downarrow \textbf{-1}}$
- $\dfrac{E\{x\} = \text{False}}{E \vdash \text{x} \Downarrow \textbf{False}}$

E-Plus

$$\frac{E \vdash x \Downarrow n \qquad E \vdash x' \Downarrow n'}{E \vdash x + x' \Downarrow n + n'}$$

Examples

- $\dfrac{E \vdash x \Downarrow 5 \quad E \vdash y \Downarrow -1}{E \vdash x + y \Downarrow \mathbf{4}}$

E-Op

$$\frac{E \vdash x \Downarrow n \quad E \vdash x' \Downarrow n'}{E \vdash x + x' \Downarrow n \ \mathbf{op} \ n'}$$

Examples

- E-Plus: $\dfrac{E \vdash x \Downarrow 5 \quad E \vdash y \Downarrow -1}{E \vdash x + y \Downarrow \mathbf{4}}$

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Examples

- E-Plus: $\dfrac{E \vdash x \Downarrow 5 \quad E \vdash y \Downarrow -1}{E \vdash x + y \Downarrow \mathbf{4}}$
- E-Or: $\dfrac{E \vdash x \Downarrow \mathsf{False} \quad E \vdash y \Downarrow \mathsf{True}}{E \vdash x \vee y \Downarrow \textbf{False}}$

E-Op

$$\frac{E \vdash x \Downarrow n \quad E \vdash x' \Downarrow n'}{E \vdash x + x' \Downarrow n \ \mathbf{op} \ n'}$$

Examples

- E-Plus: $\dfrac{E \vdash x \Downarrow 5 \quad E \vdash y \Downarrow -1}{E \vdash x + y \Downarrow \mathbf{4}}$
- E-Or: $\dfrac{E \vdash x \Downarrow \textsf{False} \quad E \vdash y \Downarrow \textsf{True}}{E \vdash x \vee y \Downarrow \textbf{False}}$
- E-NotTrue: $\dfrac{E \vdash x \Downarrow \textsf{True}}{E \vdash \neg x \Downarrow \textbf{False}}$

Applying Rules

- Form a tree of inference rules
  - top ones are axioms or directly given
  - bottom one is the conclusion, i.e., what we want to prove

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- Form a tree of inference rules
  - top ones are axioms or directly given
  - bottom one is the conclusion, i.e., what we want to prove
- Example: given $E = \{x \mapsto 3, y \mapsto 5\}$, prove $y < x + 3$ reduces to True

$$\dfrac{\dfrac{E\{y\} = 5}{E \vdash y \Downarrow 5} \qquad \dfrac{\dfrac{E\{x\} = 3}{E \vdash x \Downarrow 3} \qquad \dfrac{}{E \vdash 3 \Downarrow 3}}{E \vdash x + 3 \Downarrow 6}}{E \vdash y < x + 3 \Downarrow \textbf{True}}$$

In-class Exercise

Given $E = \{x \mapsto 3, y \mapsto 5\}$, prove $(x > y) \wedge T$ reduces to False

Rules for WHILE statements

- Previous rules are for *expressions* and reduce to values
  - E-Var gives the value of some variable
  - E-Plus gives the sum value of two variables

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- Previous rules are for *expressions* and reduce to values
  - E-Var gives the value of some variable
  - E-Plus gives the sum value of two variables
- Now, rules for statements (e.g., assignment) do not reduce to a value, but an updated environment
  - Rule for the assignment $x := n$ gives a new environment that is similar to the old one, but with x maps to n

E-Assign

$$\frac{E \vdash a \Downarrow n}{E \vdash x := a \Downarrow E\{x \mapsto n\}}$$

$E\{x \mapsto n\}$: same as E, but with $x \mapsto n$

E-Assign
$$\frac{E \vdash a \Downarrow n}{E \vdash x := a \Downarrow E\{x \mapsto n\}}$$

E-Skip
$$\frac{}{E \vdash \text{skip} \Downarrow E}$$

E-Seq
$$\frac{E \vdash S1 \Downarrow E' \quad E' \vdash S2 \Downarrow E''}{E \vdash S1; S2 \Downarrow E''}$$

E-IfTrue
$$\frac{E \vdash b \Downarrow \mathsf{True} \quad E \vdash S1 \Downarrow E'}{E \vdash \text{if } b \text{ then } S1 \text{ else } S2 \Downarrow E'}$$

E-IfFalse
$$\frac{E \vdash b \Downarrow \mathsf{False} \quad E \vdash S2 \Downarrow E''}{E \vdash \text{if } b \text{ then } S1 \text{ else } S2 \Downarrow E''}$$

E-While
$$\frac{E \vdash b \Downarrow \text{True} \quad E \vdash S; \text{while } b \text{ do } S \Downarrow \text{E'}}{E \vdash \text{while } b \text{ do } S \Downarrow E'}$$

$$\frac{E \vdash b \Downarrow \text{False}}{E \vdash \text{while } b \text{ do } S \Downarrow E}$$

WHILE Execution Example

1.	$\{\} \vdash x := 5; \text{if } x > 3 \text{ then } y := 1 \text{ else } y := 5 \Downarrow \{x \mapsto 5, y \mapsto 1\}$	E-Seq on 2, 4
2.	$\{\} \vdash x := 5 \Downarrow x \mapsto 5$	E-Assign on 3
3.	$\{\} \vdash 5 \Downarrow 5$	E-Val
4.	$\{x \mapsto 5\} \vdash \text{if } x > 3 \text{ then } y := 1 \text{ else } y := 5 \Downarrow \{x \mapsto 5, y \mapsto 1\}$	E-IfTrue 5, 8
5.	$\{x \mapsto 5\} \vdash x > 3 \Downarrow \text{True}$	E-Op 6,7
6.	$\{x \mapsto 5\} \vdash x \Downarrow 5$	E-Var
7.	$\{x \mapsto 5\} \vdash 3 \Downarrow 3$	E-Val
8.	$\{x \mapsto 5\} \vdash y := 1 \Downarrow \{x \mapsto 5, y \mapsto 1\}$	E-Assign 9
9.	$\{x \mapsto 5\} \vdash 1 \Downarrow 1\}$	E-Var

In-class Exercise

Given $E = \{\}$, prove
$x := \mathrm{T}; \mathrm{if}\ x \equiv \mathrm{T}\ \mathrm{then}\ y := 5; x := \neg x;\ \mathrm{else\ skip}$
gives $\{x \mapsto \mathrm{False}, y \mapsto 5\}$